

# ON MUTATION OF $\tau$ -TILTING MODULES

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**ABSTRACT.** Mutation of  $\tau$ -tilting modules is a basic operation to construct a new support  $\tau$ -tilting module from a given one by replacing a direct summand. The aim of this paper is to give a positive answer to the question posed in [AIR, Question 2.31] about mutation of  $\tau$ -tilting modules.

## 1. INTRODUCTION

$\tau$ -tilting theory was introduced by Adachi, Iyama and Reiten [AIR] and completes (classical) tilting theory from the viewpoint of mutation. Note that  $\tau$ -tilting theory has stimulated several investigations; in particular, there is close relation between support  $\tau$ -tilting modules (see definition 2.1 for details) and some other important notions in representation theory, such as torsion classes, silting complexes, cluster-tilting objects and  $*$ -modules (see [AIR, AiI, AMV, IJY, IR, J, W] and so on). Since  $\tau$ -tilting theory was introduced, many algebraists started to apply it to important classes of algebras (see [Ad, EJR, HZ, IZ, M1, M2, Z] and so on).

Let us recall a main result in the paper [AIR]. Let  $\Lambda$  be a finite dimensional algebra and  $T = X \oplus U$  a basic  $\tau$ -tilting  $\Lambda$ -module with an indecomposable summand  $X$  satisfying  $X \notin \text{Fac } U$ . Take an exact sequence

$$X \xrightarrow{f} U' \longrightarrow Y \longrightarrow 0 \quad (1.1)$$

with a left  $(\text{add } U)$ -approximation  $f$ . It is shown in [AIR, Theorem 2.30] that  $Y$  is either zero or a direct sum of copies of an indecomposable  $\Lambda$ -module  $Z$ , and we can obtain a new basic support  $\tau$ -tilting  $\Lambda$ -module  $\mu_X(T)$  called *mutation* of  $T$  with respect to  $X$  by  $\mu_X(T) = U$  if  $Y = 0$  and  $\mu_X(T) = Z \oplus U$  if  $Y \neq 0$ .

Moreover, they posed the following question.

**Question 1.1.** Assume that  $Y$  in (1.1) is nonzero. Is  $Y$  indecomposable?

A partial answer for the case when  $\Lambda$  is an endomorphism algebra of a cluster-tilting object was given by Yang and Zhu in [YZ, Corollary 4.17]. The aim of this paper is to give a positive answer to this question.

**Theorem 1.2.** If  $Y$  in (1.1) is nonzero, then it is indecomposable.

The idea of proof is to use the bijection between support  $\tau$ -tilting modules and two-term silting complexes given in [AIR].

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**Notation.** Let  $K$  be a field. By an algebra  $\Lambda$ , we mean a finite dimensional  $K$ -algebra. We denote by  $\text{mod } \Lambda$  (resp.  $\text{proj } \Lambda$ ) the category of finitely generated (resp. finitely generated projective) left  $\Lambda$ -modules and by  $\tau$  the Auslander-Reiten translation of  $\Lambda$ . We denote by  $K^b(\text{proj } \Lambda)$  the homotopy category of bounded complexes of finitely generated projective  $\Lambda$ -modules. For  $X \in \text{mod } \Lambda$ , we denote by  $\text{add } X$  (resp.  $\text{Fac } X$ ) the subcategory of  $\text{mod } \Lambda$  consisting of direct summands (resp. factor modules) of finite direct sums of copies of  $X$ .

## 2. PROOF OF THEOREM

First we recall the definition of support  $\tau$ -tilting modules from [AIR].

**Definition 2.1.** Let  $X \in \text{mod } \Lambda$  and  $P \in \text{proj } \Lambda$ .

- (1) We call  $X$   *$\tau$ -rigid* if  $\text{Hom}_\Lambda(X, \tau X) = 0$ . We call  $(X, P)$  a  *$\tau$ -rigid pair* if  $X$  is  $\tau$ -rigid and  $\text{Hom}_\Lambda(P, X) = 0$ .
- (2)  $X$  is called  *$\tau$ -tilting* if  $X$  is  $\tau$ -rigid and  $|X| = |\Lambda|$ .
- (3)  $X$  is called *support  $\tau$ -tilting* if there exists an idempotent  $e$  of  $\Lambda$  such that  $X$  is a  $\tau$ -tilting  $(\Lambda/\langle e \rangle)$ -module. We call  $(X, P)$  a *support  $\tau$ -tilting pair* if  $(X, P)$  is  $\tau$ -rigid and  $|X| + |P| = |\Lambda|$ .

If  $(X, P)$  is a support  $\tau$ -tilting pair for  $\Lambda$ , then  $X$  is a support  $\tau$ -tilting  $\Lambda$ -module. Conversely any support  $\tau$ -tilting  $\Lambda$ -module  $X$  can be extended to a support  $\tau$ -tilting pair  $(X, P)$ . We denote by  $s\tau\text{-tilt } \Lambda$  the set of isomorphism classes of basic support  $\tau$ -tilting pairs (or equivalently, modules) for  $\Lambda$ .

Now we recall the definition of silting complexes from [AiI].

**Definition 2.2.** We call  $P \in K^b(\text{proj } \Lambda)$  *silting* if  $\text{Hom}_{K^b(\text{proj } \Lambda)}(P, P[i]) = 0$  for any  $i > 0$  and thick  $P = K^b(\text{proj } \Lambda)$ , where thick  $P$  is the smallest full subcategory of  $K^b(\text{proj } \Lambda)$  containing  $P$  and is closed under cones,  $[\pm 1]$  and direct summands. A complex  $P = (P^i, d^i)$  in  $K^b(\text{proj } \Lambda)$  is called *two-term* if  $P^i = 0$  for all  $i \neq 0, -1$ .

We denote by  $2\text{-silt } \Lambda$  the set of isomorphism classes of basic two-term silting complexes in  $K^b(\text{proj } \Lambda)$ . Moreover, recall the definition of mutation of silting complexes from [AiI].

**Definition-Proposition 2.3.** [AiI, Theorem 2.31] Let  $P = X \oplus Q$  be a basic silting complex in  $K^b(\text{proj } \Lambda)$  with an indecomposable summand  $X$ . We take a minimal left  $(\text{add } Q)$ -approximation  $f$  and a triangle

$$X \xrightarrow{f} Q' \longrightarrow Y \longrightarrow X[1].$$

Then  $Y$  is indecomposable and  $\mu_X^-(P) := Y \oplus Q$  is a basic silting complex in  $K^b(\text{proj } \Lambda)$  called the *left mutation* of  $P$  with respect to  $X$ . The *right mutation* is defined dually but we will not use it in this paper.

The following result establishes a relation between  $2\text{-silt } \Lambda$  and  $s\tau\text{-tilt } \Lambda$ .

**Theorem 2.4.** [AIR, Theorem 3.2 and Corollary 3.9] There exists a bijection

$$2\text{-silt } \Lambda \longleftrightarrow s\tau\text{-tilt } \Lambda \quad (2.1)$$

given by  $2\text{-silt } \Lambda \ni P \mapsto H^0(P) \in s\tau\text{-tilt } \Lambda$  and  $s\tau\text{-tilt } \Lambda \ni (M, P) \mapsto (P_1 \oplus P \xrightarrow{(f,0)} P_0) \in 2\text{-silt } \Lambda$ , where  $f : P_1 \rightarrow P_0$  is a minimal projective presentation of  $M$ . Moreover, the bijection (2.1) preserves mutation.

Now we give the proof of main result of this paper.

*Proof of Theorem 1.2.* Assume  $Y \neq 0$  in (1.1).

By taking the minimal projective presentations of  $X$  and  $U$ . We have the following exact sequences:

$$P_X^{-1} \xrightarrow{d_X^{-1}} P_X^0 \xrightarrow{d_X^0} X \longrightarrow 0, \quad P_U^{-1} \xrightarrow{d_U^{-1}} P_U^0 \xrightarrow{d_U^0} U \longrightarrow 0.$$

Denote by  $P_X = (P_X^{-1} \xrightarrow{d_X^{-1}} P_X^0)$  and  $P_U = (P_U^{-1} \xrightarrow{d_U^{-1}} P_U^0)$ . Then  $P_T = P_X \oplus P_U$  gives a minimal projective presentation of  $T$ . Then by Theorem 2.4 it follows that  $P_T$  belongs to  $2\text{-silt } \Lambda$  since  $T$  is a basic  $\tau$ -tilting  $\Lambda$ -module. Also  $P_X$  is indecomposable since  $X$  is indecomposable.

Under above setting,  $\mu_X(T) \in s\tau\text{-tilt } \Lambda$  and  $\mu_{P_X}^-(P_T) \in 2\text{-silt } \Lambda$  correspond via the bijection (2.1) in Theorem 2.4. In particular, we have

$$\mu_X(T) = H^0(\mu_{P_X}^-(P_T)).$$

To calculate  $\mu_{P_X}^-(P_T)$  in  $K^b(\text{proj } \Lambda)$ , we take a triangle

$$P_X \xrightarrow{a} P' \xrightarrow{b} Q \longrightarrow P_X[1], \quad (2.2)$$

where  $a$  is a minimal left  $(\text{add } P_U)$ -approximation of  $P_X$ . Then by Definition-Proposition 2.3,  $Q$  is indecomposable and  $\mu_{P_X}^-(P_T) = Q \oplus P_U$ .

Taking the 0th cohomology of the triangle (2.2), we obtain the following exact sequence:

$$X \xrightarrow{f'} U_{P'} \xrightarrow{g'} Y_Q \longrightarrow 0,$$

where  $U_{P'} = H^0(P') \in H^0(\text{add } P_U) = \text{add } U$ ,  $Y_Q = H^0(Q)$ ,  $f' = H^0(a)$  and  $g' = H^0(b)$ . Since  $Q$  is indecomposable, it follows that  $Y_Q$  is indecomposable.

We claim that  $f'$  is a left  $(\text{add } U)$ -approximation. For any  $h \in \text{Hom}_\Lambda(X, U)$ , there exist morphisms  $c^{-1}$  and  $c^0$  making the following diagram commutative:

$$\begin{array}{ccccc} P_X^{-1} & \xrightarrow{d_X^{-1}} & P_X^0 & \xrightarrow{d_X^0} & X \longrightarrow 0 \\ \downarrow c^{-1} & & \downarrow c^0 & & \downarrow h \\ P_U^{-1} & \xrightarrow{d_U^{-1}} & P_U^0 & \xrightarrow{d_U^0} & U \longrightarrow 0. \end{array}$$

Define  $c = \overline{(c^{-1}, c^0)} \in \text{Hom}_{K^b(\text{proj } \Lambda)}(P_X, P_U)$ . Immediately, we have  $H^0(c) = h$ .

Since  $a \in \text{Hom}_{K^b(\text{proj } \Lambda)}(P_X, P')$  is a left  $(\text{add } P_U)$ -approximation, there exists  $e \in \text{Hom}_{K^b(\text{proj } \Lambda)}(P', P_U)$  such that  $c = ea \in \text{Hom}_{K^b(\text{proj } \Lambda)}(P_X, P_U)$ . Since  $H^0(-) : K^b(\text{proj } \Lambda) \rightarrow \text{mod } \Lambda$  is a functor, we have

$$h = H^0(c) = H^0(ea) = H^0(e)H^0(a) = H^0(e)f'.$$

We have finished to prove that  $f'$  is a left  $(\text{add } U)$ -approximation.

Since  $f$  is a minimal left  $(\text{add } U)$ -approximation, there exists a module  $U''$  in  $\text{add } U$  such that  $U_{P'} = U' \oplus U''$  and  $Y_Q = Y \oplus U''$ . Since  $Y_Q$  is indecomposable and  $Y \neq 0$  by our assumption, we have that  $U'' = 0$  and  $Y = Y_Q$  is indecomposable.  $\square$

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